

Consequences of the choice of a particular basis of $L^2(S^3)$ for the cubic wave equation on the sphere and the Euclidian space

Anne-Sophie de Suzzoni*

July 11, 2012

Abstract

In this paper, the almost sure global well-posedness of the cubic non linear wave equation on the sphere is studied when the initial datum is a random variable with values in low regularity spaces. The domain is first the 3D sphere, thanks to the existence of a uniformly bounded in L^p basis of $L^2(S^3)$ and then the result is extended to \mathbb{R}^3 thanks to the Penrose transform.

Contents

1	Introduction	1
2	Almost sure existence of global solutions on the sphere	4
2.1	Definition of the initial data and local theory	4
2.2	Global solutions on the sphere : case 1	11
2.3	Global solutions on the sphere : case 2	12
3	Reduction to the sphere and almost sure solutions on the Euclidian space	15
3.1	Penrose transform and reduction to the sphere	15
3.2	Properties of the change of variable	17
3.3	Spaces of definition of the initial data	19
4	Uniqueness of the solution and scattering	20
4.1	Uniqueness	20
4.2	Scattering property	23

1 Introduction

The first aim of this paper is to extend the result by N. Burq and N. Tzvetkov [2] on the torus to the sphere of dimension 3. In [2], N. Burq and N. Tzvetkov have proved the global well-posedness

*University of Cergy-Pontoise, UMR CNRS 8088, F-95000 Cergy-Pontoise, e-mail : anne-sophie.de-suzzoni@u-cergy.fr

of the cubic non linear equation when the initial datum is a randomization of some function in the product of Sobolev spaces $H^\sigma(\mathbb{T}^3) \times H^{\sigma-1}(\mathbb{T}^3)$, $\sigma \geq 0$ and \mathbb{T}^3 the torus of dimension 3.

The probabilistic estimates they use in order to prove their result are due to the fact that the L^p norms of the canonical basis of $L^2(\mathbb{T}^3) : (e^{in \cdot x})_{n \in \mathbb{N}^3}$, are uniformly bounded, whether in n or in p .

Here, a result of N. Burq of G. Lebeau is required to go on in the case of the sphere, as the basis of $L^2(S^3)$ have a priori no reason to be uniformly bounded in L^p . In [1], they proved that there exists a basis of L^2 uniformly bounded in L^p by $C\sqrt{p}$ and formed by eigen functions of the Laplace-Beltrami operator on the sphere. As one can see, the bound does depend on p , which proved to be an obstacle to extend the result of [2].

Nevertheless, let us describe the above-mentioned randomization. Let $(e_{n,k})_{n,k}$ be such a basis of $L^2(S^3)$, that is, such that for all n, k

$$\|e_{n,k}\|_{L^p} \leq C\sqrt{p},$$

and

$$-\Delta_{S^3} e_{n,k} = n^2 e_{n,k},$$

let $a_{n,k}$ and $b_{n,k}$ be two sequences of real-valued i.i.d. on a probability space (Ω, \mathcal{A}, P) and with large Gaussian deviation estimates, which means such that there exists c such that for all γ , the following mean values satisfy :

$$E(e^{\gamma a_{n,k}}), E(e^{\gamma b_{n,k}}) \leq e^{c\gamma^2},$$

and finally, let $\lambda_{n,k}$ and $\mu_{n,k}$ be two sequences of complex numbers such that

$$\sum_{n,k} (1+n^2)^\sigma |\lambda_{n,k}|^2 < \infty \text{ and } \sum_{n,k} (1+n^2)^{\sigma-1} |\mu_{n,k}|^2 < \infty.$$

Then the equation

$$(\partial_T^2 + 1 - \Delta_{S^3})u + |u|^2 u = 0$$

with initial datum the randomization of $\sum \lambda_{n,k} e_{n,k}$, $\sum \mu_{n,k} e_{n,k}$ defined as

$$u|_{T=0} = u_0 = \sum_{n,k} \lambda_{n,k} a_{n,k} e_{n,k}, \quad \partial_T u|_{T=0} = u_1 = \sum_{n,k} \mu_{n,k} b_{n,k} e_{n,k},$$

is globally well-posed.

The measure μ induced by the couple $(u_0, u_1) \in L^2(\Omega, H^\sigma \times H^{\sigma-1})$ is very similar to the one introduced by the randomization in [3].

To phrase it more precisely,

Theorem 1.1. *There exists a set E of full μ -measure such that for all $(v_0, v_1) \in E$ the Cauchy problem*

$$\begin{cases} (\partial_T^2 + 1 - \Delta_{S^3})u + |u|^2 u = 0 \\ u|_{T=0} = v_0 \end{cases} \quad \partial_T u|_{T=0} = v_1$$

has a unique solution in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1(S^3))$ where $U(T)$ is the flow of the linear equation $\partial_T^2 + 1 - \Delta_{S^3} = 0$.

Note that the wave operator $\partial_T^2 - \Delta_{S^3}$ has been replaced here by $\partial_T^2 + 1 - \Delta_{S^3}$ for convenience wrt the second topic of this paper. However the proof for the cubic non linear wave equation would be analog to the one with this operator.

What is more, if the $\lambda_{n,k}$ (resp. or the $\mu_{n,k}$) are supposed to be such that

$$\sum_{n,k} (1+n^2)^s |\lambda_{n,k}|^2 = +\infty \text{ (resp. or } \sum_{n,k} (1+n^2)^{s-1} |\mu_{n,k}|^2 = +\infty)$$

for all $s > \sigma$, and the $a_{n,k}$ and $b_{n,k}$ are complex Gaussians of law $\mathcal{N}(0, 1)$, then the elements of E are almost surely in $H^\sigma \times H^{\sigma-1}$ and almost surely not in $H^s \times H^{s-1}$.

As it appears, this result recalls one of [5] on the sphere but without the hypothesis of radial symmetry.

The main idea behind the proof is that with large Gaussian deviation estimates, the solution of the linear equation $U(T)(u_0, u_1)$ is made to belong almost surely to L^p for all $p \geq 1$ which ensures local and then global well-posedness. Indeed, it is the gain on integrability on the initial data that helps to gain regularity on the non linear part (namely, the solution minus $U(T)(u_0, u_1)$) of the solution.

A second issue raised on this paper is the properties of the Penrose transform of the solution. The Penrose transform sends solutions of $(\partial_T^2 + 1 - \Delta_{S^3})u + |u|^2 u = 0$ on the sphere to solution of the cubic non linear wave equation on the Euclidian space \mathbb{R}^3 . Indeed, the change of variable involved in this transform injects $\mathbb{R} \times \mathbb{R}^3$ into $[-\pi, \pi] \times S^3$ and satisfies nice properties wrt the d'Alembertian $\partial_t^2 - \Delta_{\mathbb{R}^3}$.

Hence, with a solution of $(\partial_T^2 + 1 - \Delta_{S^3})u + |u|^2 u = 0$ on the sphere, the existence of a solution of the cubic NLW on \mathbb{R}^3 is expected.

Nevertheless, the use of the Penrose transform raises three problems : first, the space where the solution lives shall be described, then, so does the space where this solution is unique, and finally, the spaces to which the initial data belong or do not belong should be specified.

Unfortunately, the third matter remains a wonder but the author believes that if the work on the sphere is done with $\sigma = 0$, that is with the initial data on the sphere in $L^2 \times H^{-1}$, then the initial data on \mathbb{R}^3 should be almost surely in $L^2 \times H^{-1}$ when multiplied by $((\frac{2}{1+r^2})^{1/2}, ((\frac{2}{1+r^2})^{-1/2})$, and almost surely not to $H^s \times H^{s-1}$, when $s > 0$.

However, the following theorem holds.

Theorem 1.2. *There exists a measure ν on $\mathcal{L}^2(\mathbb{R}^3) \times \mathcal{H}^{-1}(\mathbb{R}^3)$ with*

$$\|g\|_{\mathcal{L}^2} = \left\| \left(\frac{2}{1+r^2} \right)^{1/2} g \right\|_{L^2}, \quad \|g\|_{\mathcal{H}^{-1}(\mathbb{R}^3)} = \left\| \left(\frac{2}{1+r^2} \right)^{-1/2} (1 - H_1)^{-1} g \right\|_{L^2}$$

and

$$H_1 = \left(\frac{1+r^2}{2} \right)^2 \Delta_{\mathbb{R}^3} + 3 \frac{1+r^2}{2} r \partial_r + 6 \frac{1+r^2}{2}$$

and a set F of full ν -measure such that for all $(g_0, g_1) \in F$, the Cauchy problem :

$$\begin{cases} (\partial_t^2 - \Delta_{\mathbb{R}^3})f + |f|^2 f = 0 \\ f|_{t=0} = g_0 & \partial_t f|_{t=0} = g_1 \end{cases}$$

has a unique global solution in $L(t)(g_0, g_1) + C(\mathbb{R}, H^1(\mathbb{R}^3))$ where $L(t)$ is the flow of the linear wave equation.

Moreover, the solution f satisfies scattering in the sense that for all $q \in]\frac{18}{5}, 6]$,

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} = O((1 + t^2)^{-1/6})$$

when $t \rightarrow \pm\infty$.

The main idea of the proof is that for $q \geq 4$, the L^q norm of the solution on \mathbb{R}^3 is controlled by the L^q norm of the solution on the sphere, which ensures regularity properties and then uniqueness of the solution.

Plan of the paper The section 2 is dedicated to the proof of the global well-posedness of the equation on the sphere. The first two subsections, which are about local theory and global theory on $H^\sigma \times H^{\sigma-1}$ with $\sigma > 0$, are very similar to [2]. The third one, where the global well-posedness is dealt with the initial datum being almost surely in $L^2 \times H^{-1}$ presents divergences, in particular due to the fact that the bound on the L^p norms of the chosen basis of L^2 depends on p .

The third section is about the Penrose transform and how it acts on the norms of the solution and the norms of the initial data. The transform is presented, along with its trace on the initial data, and then what its trace turns the Laplace-Beltrami operator on the sphere to in order to study the Sobolev norms of the initial data on \mathbb{R}^3 .

The fourth one is about the uniqueness and scattering properties of the solution of the equation on \mathbb{R}^3 . It focuses on the integrability of the solution (how a L^p norm of the solution on the sphere is changed by the Penrose transform), then its regularity before stating the uniqueness and scattering results.

Acknowledgments

The author would like to thank Nicolas Burq for suggesting the problem.

2 Almost sure existence of global solutions on the sphere

This section deals with the global well-posedness for the cubic wave equation on the sphere with initial data taken as a random variable on $H^s \times H^{s-1}$, $s \geq 0$. The solution has a linear part in $C(\mathbb{R}, H^s)$ and a non linear part in $C(\mathbb{R}, H^1)$.

2.1 Definition of the initial data and local theory

Following the techniques of N. Burq and N. Tzvetkov in [2], the random initial datum shall be chosen such that when it is submitted to the linear flow of the wave equation, it has L^p norms in time and space.

The theorem 6 of the third section of [1] provides the existence of a hilbertian basis of $L^2(S^3)$ composed with spherical harmonics uniformly bounded in L^p . Let us therefore name the different objects that shall be needed to define a suitable initial datum.

First, let us recall that the eigenvalues of $1 - \Delta_{S^3}$ are n^2 , $n \geq 1$.

Thanks to the result of N. Burq and G. Lebeau, denote by $(e_{n,k})_{n \geq 1, 1 \leq k \leq (n+1)^2}$ a Hilbertian basis of $L^2(S^3)$ uniformly bounded in L^p .

Theorem 2.1 (Burq, Lebeau). *There exists a Hilbertian basis $(e_{n,k})_{n,k}$ of $L^2(S^3)$ such that :*

$$(1 - \Delta_{S^3})e_{n,k} = n^2 e_{n,k}$$

for all $n \geq 1, 1 \leq k \leq (n+1)^2$, $(n+1)^2$ being the dimension of the subspace of L^2 spanned by the spherical harmonics of degree $n+1$, and such that there exists a constant C such that for all n, k

$$\|e_{n,k}\|_{L^p(S^3)} \leq C \sqrt{p}.$$

As afore-mentioned, the main difference between this section of this paper and the one by Burq and Tzvetkov [2] is that in their paper, the basis of L^2 is bounded uniformly in L^p , but uniformly in terms of p too. This property allows them to ask for an almost sure L^p bound (so to speak) on the initial datum and take $p \rightarrow \infty$. The difference will appear and be detailed later.

Let $\sigma \in [0, \frac{1}{2}[$ and $(u_0^{n,k})_{n,k}$ and $(u_1^{n,k})_{n,k}$ be two sequences of real numbers such that the series

$$\sum_{n \geq 1} n^{2\sigma} \sum_{1 \leq k \leq (n+1)^2} (u_0^{n,k})^2 \text{ and } \sum_{n \geq 1} n^{2(\sigma-1)} \sum_{1 \leq k \leq (n+1)^2} (u_1^{n,k})^2$$

converge but at least one of the series

$$\sum_{n \geq 1} n \sum_{1 \leq k \leq (n+1)^2} (u_0^{n,k})^2 \text{ or } \sum_{n \geq 1} n^{-1} \sum_{1 \leq k \leq (n+1)^2} (u_1^{n,k})^2$$

diverge, that is to say

$$\left(\sum_{n,k} u_0^{n,k} e_{n,k}, \sum_{n,k} u_1^{n,k} e_{n,k} \right)$$

belongs to $H^\sigma \times H^{\sigma-1}$ but is not in the critical space for the cubic NLW $H^{1/2} \times H^{-1/2}$.

Let (X, \mathcal{A}, P) be a probability space large enough such that two sequences $(a_{n,k})$ and $(b_{n,k})$ of random variables can be taken satisfying that the $a_{n,k}$ are independent from each other and from the $b_{n,k}$, the $b_{n,k}$ are independent from each other and that there exists a such that for all n, k and all $\gamma \in \mathbb{R}$:

$$E(e^{\gamma a_{n,k}}), E(e^{\gamma b_{n,k}}) \leq e^{a\gamma^2}$$

where E is the mean value wrt the probability measure P , which ensures that the random variables have Gaussian-like large deviation estimates.

Proposition 2.2. *The sequences of $L^2(X, H^\sigma(S^3))$ and $L^2(X, H^{\sigma-1}(S^3))$ respectively*

$$u_0^N = \sum_{n=1}^N \sum_{k=1}^{(n+1)^2} u_0^{n,k} a_{n,k} e_{n,k} \text{ and } u_1^N = \sum_{n=1}^N \sum_{k=1}^{(n+1)^2} u_1^{n,k} b_{n,k} e_{n,k}$$

converges. Let u_0 and u_1 their limits.

Proof. The proof consists in the fact that the mean values of $a_{n,k}^2$ and $b_{n,k}^2$ are uniformly bounded by $8a$. It ensures that the sequences are Cauchy in their respective spaces and therefore that they converge. \square

Let $U(T)$ be the flow of the linear equation $(\partial_T^2 + 1 - \Delta_{S^3})u = 0$, that is

$$U(T) \left(\sum_{n,k} v_0^{n,k} e_{n,k}, \sum_{n,k} v_1^{n,k} e_{n,k} \right) = \sum_{n,k} \left(\cos(nT) v_0^{n,k} + \frac{\sin(nT)}{n} v_1^{n,k} \right) e_{n,k}.$$

Set

$$S_M^N = \sum_{n=N}^M \sum_k n^{2\sigma} (u_0^{n,k})^2 + n^{2(\sigma-1)} (u_1^{n,k})^2,$$

$$S_M = S_M^0, S^N = \lim_{M \rightarrow \infty} S_M^N \text{ and } S = S_N + S^N.$$

Set also $\Pi_N, N \geq 0$ the orthogonal projection on the subspace of L^2 spanned by $\{e_{n,k} \mid n \leq N\}$ with the convention $\Pi_0 = 0$.

The initial data u_0, u_1 satisfy some properties regarding the spaces where they belong :

Proposition 2.3. *There exists $C, c > 0$ such that for all $\lambda \geq 0$:*

- for all $N \geq 0$, all $p \geq 1$ and with $\delta_p = \frac{2}{p} > \frac{1}{p}$

$$P(\| \frac{1}{1 + |T|^{\delta_p}} (1 - \Pi_N) U(T)(u_0, u_1) \|_{L^p(\mathbb{R} \times S^3)} \geq \lambda) \leq \left(\frac{Cp \sqrt{S^N}}{\lambda} \right)^p,$$

- with $\delta_3 = 2/3 > 1/3$,

$$P(\| \frac{1}{1 + |T|^{\delta_3}} U(T)(u_0, u_1) \|_{L_T^3, L^6(S^3)} \geq \lambda) \leq C e^{-c\lambda^2/S}$$

- for all $M \geq 1$ and with $s = 1$ if $\sigma = 0$ and $s = 0$ otherwise

$$P(\| \frac{1}{1 + T^2} \Pi_M U(T)(u_0, u_1) \|_{L_T^1, L^\infty(S^3)} \geq \lambda) \leq C e^{-c\lambda^2/(M^s S)}.$$

Remark 2.1. *This proposition differs from the analog one in the torus case where the first inequality corresponded to :*

$$P(\| \frac{1}{1 + |T|^{\delta_p}} (1 - \Pi_N) U(T)(u_0, u_1) \|_{L^p(\mathbb{R} \times S^3)} \geq \lambda) \leq \left(\frac{C \sqrt{p} \sqrt{S^N}}{\lambda} \right)^p.$$

Proof.

Lemma 2.4. *There exists C such that for all $q \geq 1$ and all couple of l^2 sequences $v_{n,k}, w_{n,k}$:*

$$\| \sum_{n,k} a_{n,k} v_{n,k} + b_{n,k} w_{n,k} \|_{L_X^q} \leq C \sqrt{q} \left(\sum |v_{n,k}|^2 + |w_{n,k}|^2 \right)^{1/2}.$$

The proof can be found in [2]

Lemma 2.5. *There exists C such that for all $r, p \geq 1$, $s \geq 0$, $M > N \geq 0$ and $q \geq r, p$,*

$$\left\| \frac{1}{1 + |T|^{2/r}} (1 - \Pi_N)(1 - \Delta)^{s/2} U(T)(u_0^M, u_1^M) \right\|_{L_X^q, L_T^r, L^p(S^3)} \leq C \sqrt{p} \sqrt{q} M^{s'} \sqrt{S_M - S_N}$$

with $s' = s - \sigma$ if $s \geq \sigma$ and $s' = 0$ otherwise.

Proof. Let

$$\Sigma_N^M(x) = \sum_{n=N+1}^M \sum_{k=1}^{1+n^2} n^{2s} \left((u_0^{n,k})^2 + n^{-2} (u_1^{n,k})^2 \right) |e_{n,k}(x)|^2.$$

Using the previous lemma and bounding the sines and cosines by 1 in

$$(1 - \Delta)^{s/2} U(T)(u_0^M, u_1^M),$$

it appears that

$$\left\| \frac{1}{1 + |T|^{2/r}} (1 - \Pi_N)(1 - \Delta)^{s/2} U(T)(u_0^M, u_1^M) \right\|_{L_X^q} \leq \frac{C}{1 + |T|^{2/r}} \sqrt{q} \sqrt{\Sigma_N^M(x)}.$$

Hence, as $q \geq r, p$, and thanks to Minkowski inequality, one can reverse the order of the norms

:

$$\begin{aligned} & \left\| \frac{1}{1 + |T|^{2/r}} (1 - \Pi_N)(1 - \Delta)^{s/2} U(T)(u_0^M, u_1^M) \right\|_{L_X^q, L_T^r, L^p(S^3)} \\ & \leq \left\| \frac{1}{1 + |T|^{2/r}} (1 - \Pi_N)(1 - \Delta)^{s/2} U(T)(u_0^M, u_1^M) \right\|_{L_T^r, L^p(S^3), L_X^q} \\ & \leq C \sqrt{q} \left\| \frac{1}{1 + |T|^{2/r}} \right\|_{L_T^r} \|\Sigma_N^M\|_{L^{p/2}}^{1/2}. \end{aligned}$$

The map $\frac{1}{1 + |T|^{2/r}}$ is in L^r and its norm is less than some constant which does not depend on r and

$$\begin{aligned} \|\Sigma_N^M\|_{L^{p/2}} & \leq \sum_{n=N+1}^M \sum_{k=1}^{1+n^2} n^{2s} \left((u_0^{n,k})^2 + n^{-2} (u_1^{n,k})^2 \right) \|e_{n,k}(x)\|_{L^p}^2 \\ & \leq Cp \sum_{n=N+1}^M \sum_{k=1}^{1+n^2} n^{2s} \left((u_0^{n,k})^2 + n^{-2} (u_1^{n,k})^2 \right) \leq Cp M^{2s'} (S_M - S_N). \end{aligned}$$

Therefore,

$$\left\| \frac{1}{1 + |T|^{2/r}} (1 - \Pi_N)(1 - \Delta)^{s/2} U(T)(u_0^M, u_1^M) \right\|_{L_X^q, L_T^r, L^p(S^3)} \leq C \sqrt{p} \sqrt{q} M^{s'} \sqrt{S_M - S_N}.$$

□

To prove the first inequality of the proposition, take $M \rightarrow \infty$, $s = 0$, and $r = q = p$ in the previous lemma to get :

$$\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0, u_1)\|_{L_X^p, L_T^p, L^p(S^3)} \leq Cp\sqrt{S^N}.$$

Then,

$$\begin{aligned} & P(\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0, u_1)\|_{L_T^p, L^p(S^3)} \geq \lambda) \\ &= P(\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0, u_1)\|_{L_T^p, L^p(S^3)}^p \geq \lambda^p) \\ &\leq \lambda^{-p} E(\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0, u_1)\|_{L_T^p, L^p(S^3)}^p) \\ &= \lambda^{-p} \|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0, u_1)\|_{L_X^p, L_T^p, L^p(S^3)}^p \\ &\leq \left(\frac{Cp\sqrt{S^N}}{\lambda} \right)^p. \end{aligned}$$

To prove the second, use the previous lemma with $r = 3$, $p = 6$, $q \geq 6$, $s = 0$, $M \rightarrow \infty$, $N = 0$ to get :

$$P(\|\frac{1}{1+|T|^{2/3}}U(T)(u_0, u_1)\|_{L_T^3, L^6(S^3)} \geq \lambda) \leq \left(\frac{C\sqrt{q}\sqrt{S}}{\lambda} \right)^q.$$

For $\lambda \geq \sqrt{6}\sqrt{S}Ce$, choose

$$q = \frac{\lambda^2}{C^2e^2S} \geq 6$$

to get

$$P(\|\frac{1}{1+|T|^{2/3}}U(T)(u_0, u_1)\|_{L_T^3, L^6(S^3)} \geq \lambda) \leq e^{-p} = e^{-c\lambda^2/S}$$

and for small λ use the fact that the probability is bounded by 1 which is less than $e^6e^{-c\lambda^2/S}$ to get for all λ

$$P(\|\frac{1}{1+|T|^{2/3}}U(T)(u_0, u_1)\|_{L_T^3, L^6(S^3)} \geq \lambda) \leq Ce^{-c\lambda^2/S}.$$

For the third inequality with $\sigma = 0$, use the previous lemma with $N = 0$, $r = 1$, $s = \frac{1}{2}$, $p = 7 > 6$, $q \geq 7$ to get

$$\begin{aligned} & \|\frac{1}{1+T^2}\Pi_M U(T)(u_0, u_1)\|_{L^q, L_T^1, L^\infty(S^3)} \\ &\leq \|\frac{1}{1+T^2}\Pi_M U(T)(u_0, u_1)\|_{L^q, L_T^1, W^{1/2,7}(S^3)} \leq CM^{1/2}\sqrt{q}\sqrt{S_M} \leq CM^{1/2}\sqrt{q}\sqrt{S} \end{aligned}$$

thanks in particular to the Sobolev embedding $W^{1/2,7} \rightarrow L^\infty$ and then

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0, u_1)\|_{L_T^1, L^\infty(S^3)} \geq \lambda) \leq \left(\frac{C \sqrt{q} M^{1/2} \sqrt{S}}{\lambda} \right)^q$$

and finally

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0, u_1)\|_{L_T^1, L^\infty(S^3)} \geq \lambda) \leq C e^{-c\lambda^2/(MS)}.$$

For the third inequality with $\sigma > 0$, use the previous lemma with $N = 0$, $r = 1$, $s = \sigma$, $p = \frac{4}{\sigma}$, $q \geq p$ to get

$$\begin{aligned} & \|\frac{1}{1+T^2}\Pi_M U(T)(u_0, u_1)\|_{L^q, L_T^1, L^\infty(S^3)} \\ & \leq \|\frac{1}{1+T^2}\Pi_M U(T)(u_0, u_1)\|_{L^q, L_T^1, W^{1/2,7}(S^3)} \leq C \sqrt{q} \sqrt{S_M} \leq C \sqrt{q} \sqrt{S}, \end{aligned}$$

then

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0, u_1)\|_{L_T^1, L^\infty(S^3)} \geq \lambda) \leq \left(\frac{C \sqrt{q} \sqrt{S}}{\lambda} \right)^q,$$

and finally, with an appropriate choice for q ,

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0, u_1)\|_{L_T^1, L^\infty(S^3)} \geq \lambda) \leq C e^{-c\lambda^2/S}.$$

□

Thanks to previous proposition, it is known now that $\frac{1}{1+|T|^{2/3}}U(T)(u_0, u_1)$ belongs almost surely to $L_T^3, L^6(S^3)$. Let us use this property in the local theory.

First, rewrite the equation on the sphere in a more convenient way.

Lemma 2.6. *The map u solves*

$$\begin{cases} \partial_T^2 u + (1 - \Delta)u + u^3 = 0 \\ u|_{T=0} = v_0 \end{cases} \quad \partial_T u|_{T=0} = v_1 \quad (1)$$

if and only if $v = u - U(T)(v_0, v_1)$ solves, with $g(T) = U(T)(v_0, v_1)$:

$$\begin{cases} \partial_T^2 v + (1 - \Delta)v + (g + v)^3 = 0 \\ v|_{T=0} = 0 \end{cases} \quad \partial_T v|_{T=0} = 0 \quad (2)$$

Proposition 2.7. *There exists C such that for all $\Lambda > 0$, all $T_0 \in \mathbb{R}$ and all g, v_0, v_1 such that*

$$\|\frac{1}{1+|T|^{2/3}}g\|_{L_T^3, L^6(S^3)}^3 \leq \Lambda, \quad \|v_0\|_{H^1} \leq \Lambda, \quad \|v_1\|_{L^2} \leq \Lambda,$$

the equation

$$\begin{cases} \partial_T^2 v + (1 - \Delta)v + (g + v)^3 = 0 \\ v|_{T=T_0} = v_0 \end{cases} \quad \partial_T v|_{T=T_0} = v_1 \quad (3)$$

has a unique solution in $C([T_0 - T_1, T_0 + T_1], H^1)$ with $T_1 = \min(1, \frac{1}{C\Lambda^2(1+T_0^2)^3})$.

Proof. Let

$$\phi_{g,v_0,v_1}(v)(T) = S(T - T_0)(v_0, v_1) - \int_{T_0}^T \frac{\sin((T - \tau)\sqrt{1 - \Delta})}{\sqrt{1 - \Delta}} \left((g + v)^3(\tau) \right) d\tau.$$

The equation (3) can be rewritten as the fixed point problem $\phi_{g,v_0,v_1}(v) = v$. The map ϕ_{g,v_0,v_1} satisfies :

$$\|\phi_{g,v_0,v_1}(v)(T)\|_{H^1} \leq C\Lambda + \int_{T_0}^T \|(g + v)^3\|_{L^2}$$

$$\|(g + v)^3\|_{L^2} = \|g + v\|_{L^6}^3 \leq C(\|g\|_{L^6}^3 + \|v\|_{L^6}^3) \leq C(\|g\|_{L^6}^3 + \|v\|_{H^1}^3)$$

$$\|\phi_{g,v_0,v_1}(v)(T)\|_{H^1} \leq C \left(\Lambda + (1 + |T - T_0|^2 + |T_0|^2) \left\| \frac{1}{1 + |\tau|^{2/3}} g \right\|_{L^3_\tau, L^6(S^3)} + \int_{T_0}^T \|v(\tau)\|_{H^1}^3 d\tau \right).$$

With $T \in [T_0 - T_1, T_0 + T_1]$,

$$\|\phi_{g,v_0,v_1}(v)\|_{L_T^\infty, H^1} \leq C \left((2 + T_0^2 + T_1^2) \Lambda + |T_1| \|v\|_{L_T^\infty, H^1} \right).$$

If $T_1 \leq \min(1, \frac{1}{C^3 \Lambda^2 (4 + T_0^2)^3})$ and $\|v\|_{L^\infty, H^1} \leq C\Lambda(4 + T_0^2)$, then

$$\|\phi_{g,v_0,v_1}(v)\|_{L_T^\infty, H^1} \leq C(4 + T_0^2)\Lambda$$

so the ball of radius $C(4 + T_0^2)\Lambda$ in $C([T_0 - T_1, T_0 + T_1], H^1(S^3))$ is stable under ϕ_{g,v_0,v_1} .

What is more, in this ball

$$\|\phi_{g,v_0,v_1}(v) - \phi_{g,v_0,v_1}(w)\|_{L_T^\infty, H^1}$$

$$\leq C\|v - w\|_{L_T^\infty, H^1} \left((2 + T_0^2) \left\| \frac{1}{1 + |\tau|^{2/3}} g \right\|_{L^3_\tau, L^6}^2 \|1_{[T_0 - T_1, T_0 + T_1]}\|_{L^3_T} + T_1 (\|v\|_{L^\infty, H^1}^2 + \|w\|_{L_T^\infty, H^1}^2) \right)$$

$$\leq C\|v - w\|_{L_T^\infty, H^1} \left(T_1^{1/3} (2 + T_0^2) \Lambda^{2/3} + T_1 \Lambda^2 (4 + T_0^2)^2 \right).$$

Therefore with C large enough and $T_1 \leq \frac{1}{C\Lambda^2(1 + T_0^2)}$, the fixed point theorem applies which concludes the proof. \square

Thanks to the local Cauchy theory, one can see that the solution of (2) can be extended for bigger times as long as the energy :

$$\mathcal{E}(T) = \int v(1 - \Delta)v + \int (\partial_T v)^2 + \frac{1}{2} \int v^4$$

is finite.

To bound this quantity, different arguments are used depending on whether the initial data have been built with $\sigma = 0$ or $\sigma > 0$.

2.2 Global solutions on the sphere : case 1

Theorem 2.8. *Suppose that $\sigma > 0$. There exists a set $E_\sigma \subseteq H^\sigma \times H^{\sigma-1}$ such that the probability*

$$P((u_0, u_1) \in E_\sigma) = 1$$

and that for all $v_0, v_1 \in E_\sigma$, the Cauchy problem (1) with initial datum v_0, v_1 is globally well-posed in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1)$.

Proof. The third inequality of the proposition 2.3 ensures that, when $\sigma > 0$, $\frac{1}{1+T^2}U(T)(u_0, u_1)$ belongs almost surely to $L_T^1, L^\infty(S^3)$ and $\frac{1}{1+|T|^{2/3}}U(T)(u_0, u_1)$ belongs almost surely to $L_T^3, L^6(S^3)$. Therefore, take for E_σ the set of initial data which satisfy

$$\left\| \frac{1}{1+T^2} U(T)(v_0, v_1) \right\|_{L_T^1, L^\infty(S^3)} < \infty ,$$

$$\left\| \frac{1}{1+|T|^{2/3}} U(T)(v_0, v_1) \right\|_{L_T^3, L^6(S^3)} < \infty .$$

For $v_0, v_1 \in E_\sigma$, call $g(T) = U(T)(v_0, v_1)$ and let v be the local solution of

$$\partial_T^2 v + (1 - \Delta)v + (g + v)^3 = 0$$

with initial datum 0, 0.

According to the local Cauchy theory, the solution v exists as long as

$$\int (\partial_T v)^2 + \int v(1 - \Delta)v$$

is finite.

Take

$$\mathcal{E}^2(T) = \int (\partial_T v)^2 + \int v(1 - \Delta)v + \frac{1}{2} \int v^4$$

and differentiate this quantity wrt T .

$$\begin{aligned} (d_T \mathcal{E}) \mathcal{E} &= \int \partial_T v \partial_T^2 v + \int \partial_T v (1 - \Delta)v + \int \partial_T v v^3 \\ &= \int (\partial_T v) (v^3 - (g + v)^3) . \end{aligned}$$

Hence,

$$d_T \mathcal{E} \leq \|v^3 - (g + v)^3\|_{L^2} \leq C \left(\|g(T)^3\|_{L^2} + \|g^2 v(T)\|_{L^2} + \|g v^2\|_{L^2} \right)$$

$$|d_T \mathcal{E}| \leq C \left(\|g(T)\|_{L^6}^3 + \|g\|_{L^6}^2 \|v\|_{L^6} + \|g\|_{L^\infty} \|v\|_{L^4}^2 \right)$$

$$|d_T \mathcal{E}| \leq C \left(\|g(T)\|_{L^6}^3 + \|g\|_{L^6}^2 \mathcal{E} + \|g\|_{L^\infty} \mathcal{E} \right)$$

thanks to Sobolev embedding $H^1 \rightarrow L^6$, and applying Gronwall lemma :

$$\mathcal{E}(T) \leq C \int_0^T \|g(\tau)\|_{L^6}^3 d\tau e^{C \int_0^T (\|g(\tau)\|_{L^6}^2 + \|g(\tau)\|_{L^\infty}) d\tau} < \infty .$$

Hence the result. \square

2.3 Global solutions on the sphere : case 2

Proposition 2.9. *Let $T_0 > 0$. There exists $C(T_0)$ such that for all $\theta > 0$ and $p = \frac{6}{\theta}$, supposing that $g = U(T)(v_0, v_1)$ can be written $g = g_1 + g_2$ with*

$$C(T_0) \left\| \frac{1}{1+T^{2/3}} g \right\|_{L_T^3, L_x^6}^3 \leq e^{p/18} \text{ and } C(T_0) \left(\left\| \frac{1}{1+T^{2/3}} g \right\|_{L_T^3, L_x^6}^2 + \left\| \frac{1}{1+T^2} g_1 \right\|_{L_T^2, L_x^\infty}^2 + \left\| \frac{1}{1+T^{2/p}} g_2 \right\|_{L_T^p, L_x^p} \right) \leq \frac{p}{18}$$

then the equation (3) has a unique solution onto $C([-T_0, T_0], H^1)$. The constant C depends on T_0 but is independent of θ .

Proof. Call

$$\mathcal{E}(v)^2 = \int v(1 - \Delta)v + \int (\partial_T v)^2 + \frac{1}{2} \int v^4.$$

If v is the local solution of (3), on its interval of definition, it comes :

$$d_T \mathcal{E}(v) \mathcal{E}(v) = \int (\partial_T v) (g^3 + 3g^2 v + 3(g_1 + g_2)v^2)$$

$$d_T \mathcal{E}(v) \leq \|g(T)\|_{L_x^6}^3 + 3\|g(T)\|_{L_x^6}^2 \|v\|_{L^6} + 3\|g_1(T)\|_{L^\infty} \|v^2\|_{L^2} + 3\|g_2(T)\|_{L^p} \|v^2\|_{L^{p'}}$$

with $1/p' + 1/p = 1/2$.

Then,

$$\|v\|_{L^6} \leq C\|v\|_{H^1} \leq C\mathcal{E}$$

$$\|v^2\|_{L^2} = \|v\|_{L^4}^2 \leq C\mathcal{E}$$

$$\|v^2\|_{L^{p'}} = \|v\|_{L^{2p'}}^2 \leq (\|v\|_{L^4}^{1-\theta} \|v\|_{L^6}^\theta)^2 \leq C\mathcal{E}^{1-\theta} \|v\|_{H^1}^{2\theta} \leq C\mathcal{E}(v)^{1+\theta}.$$

Thus,

$$d_T \mathcal{E}(v) \leq \|g(T)\|_{L_x^6}^3 + C \left((\|g(T)\|_{L_x^6}^2 + \|g_1(T)\|_{L^\infty}) \mathcal{E} + \|g_2(T)\|_{L^p} \mathcal{E}^{1+\theta} \right).$$

As \mathcal{E} is continuous and initially 0, suppose that until time T_1 it is less than $e^{p/6} = e^{1/\theta}$, then until time T_1 , it appears that :

$$d_T \mathcal{E}(v) \leq \|g(T)\|_{L_x^6}^3 + C \left((\|g(T)\|_{L_x^6}^2 + \|g_1(T)\|_{L^\infty}) + \|g_2(T)\|_{L^p} \right) \mathcal{E}.$$

Using Gronwall lemma,

$$\begin{aligned} \mathcal{E}(v) &\leq C(1 + T_0^2) \|g\|_{L_T^3, L_x^6}^3 e^{C((1+T_0^2)^{2/3} \left\| \frac{1}{1+T^{2/3}} g \right\|_{L_T^3, L_x^6}^2 + (1+T_0^2) \left\| \frac{1}{1+T^2} g_1 \right\|_{L_T^1, L^\infty} + (1+T_0^2)^{(1+p)/2p} \left\| \frac{1}{1+T^{2/p}} g_2 \right\|_{L_T^p, L_x^p})} \\ &\leq C(1 + T_0^2) \|g\|_{L_T^3, L_x^6}^3 e^{C(1+T_0^2) \left(\left\| \frac{1}{1+T^{2/3}} g \right\|_{L_T^3, L_x^6}^2 + \left\| \frac{1}{1+T^2} g_1 \right\|_{L_T^1, L^\infty} + \left\| \frac{1}{1+T^{2/p}} g_2 \right\|_{L_T^p, L_x^p} \right)}. \end{aligned}$$

Choosing $C(T_0) = C(1 + T_0^2)$, by hypothesis :

$$\mathcal{E}(v) \leq e^{p/9} < e^{p/6} .$$

Suppose that the solution v is not well posed on $[-T_0, T_0]$, then as $\mathcal{E}(v)$ controls the H^1 norm of v and the L^2 norm of $\partial_T v$, there exists a time T_1 such that for all time T smaller than T_1 , the energy $\mathcal{E}(v)$ is smaller than $e^{p/6}$ and a ϵ such that for all $T \in]T_1, T_1 + \epsilon[$, $\mathcal{E}(v) > e^{p/6}$. Then, thanks to the previous computation, until T_1 , the energy is strictly less than $e^{p/6}$ and as it is continuous, there exists ϵ' such that the energy remains smaller than $e^{p/6}$ until $T_1 + \epsilon'$ with contradicts the hypothesis.

Hence, the equation (2) has a unique solution in $C([-T_0, T_0], H^1)$ provided that g satisfies the right properties. \square

Definition 2.10. Let $\theta \in]0, 1[$, $p = \frac{6}{\theta}$ and $N(T_0)$ such that $\sqrt{S^{N(T_0)}}$ is smaller than $\frac{1}{54eC(T_0)C_1}$, where C_1 is the constant involved in the first inequality of proposition 2.3 and $C(T_0)$ is the one involved in the proposition (2.9), let

$$F_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)(v_0, v_1)\|_{L_T^3, L_x^6}^3 \leq e^{p/18}\} ,$$

$$G_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)(v_0, v_1)\|_{L_T^2, L_x^6}^2 \leq \frac{p}{54}\} ,$$

$$H_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)\Pi_N(v_0, v_1)\|_{L_T^1, L_x^\infty} \leq \frac{p}{54}\} ,$$

$$I_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)(1 - \Pi_N)(v_0, v_1)\|_{L_{T,x}^p} \leq \frac{p}{54}\} ,$$

$$J_\theta(T_0) = F_\theta \cap G_\theta \cap H_\theta \cap I_\theta .$$

Call then

$$E(T_0) = \bigcup_{\theta \in]0, 1[} J_\theta .$$

Remark 2.2. The separation between the high and low frequencies is useful there, as S^N can be taken as small as one wants and ensures that the measure of I_θ^c is small enough.

Proposition 2.11. The set $E(T_0)$ is of full μ -measure.

Proof. The measures of the complementary of the different sets defined satisfy :

$$\mu(F_\theta^c) = \mu\left(\{v_0, v_1 \mid \|U(T)(v_0, v_1)\|_{L_T^3, L_x^6}^3 > e^{p/54}\}\right) \leq Ce^{-c(T_0)}e^{p/27}$$

$$\mu(G_\theta^c) \leq \mu\left(\{v_0, v_1 \mid \|U(T)(v_0, v_1)\|_{L_T^2, L_x^6}^2 > \sqrt{\frac{p}{54C}}\}\right) \leq Ce^{-c(T_0)p}$$

$$\mu(H_\theta^c) = \mu\left(\{v_0, v_1 \mid \|U(T)\Pi_N(v_0, v_1)\|_{L_T^1, L_x^\infty} > \frac{p}{54C}\}\right) \leq Ce^{-c(T_0)p^2/N^s}$$

$$\mu(I_\theta^c) = \mu\left(\{v_0, v_1 \mid \|U(T)(1 - \Pi_N)(v_0, v_1)\|_{L_{T,x}^p} > \frac{p}{54C}\}\right) \leq \left(\frac{C_1 p 54 C(T_0) \sqrt{S^N}}{p}\right)^p$$

$$\mu(I_\theta^c) \leq e^{-p} .$$

It comes :

$$\mu(J_\theta^c) \leq C e^{-c(T_0)p} .$$

Thus, for all θ , $E(T_0)$ satisfies

$$\mu(E^c(T_0)) \leq \mu(J_\theta^c) \leq C e^{-c6/\theta} .$$

Taking the limit when θ goes to 0 :

$$\mu(E^c(T_0)) = 0 , \mu(E(T_0)) = 1 .$$

□

Proposition 2.12. *For all $(v_0, v_1) \in E(T_0)$, the cubic non linear wave equation on the sphere (1) with initial datum v_0, v_1 has a unique solution in $U(T)(v_0, v_1) + C([-T_0, T_0], H^1)$.*

Proof. The equation (2) with $g = U(T)(v_0, v_1) = g_1 + g_2$, $g_1 = \Pi_N g$, $g_2 = (1 - \Pi_N)g$ is equivalent to (1) and satisfies the hypothesis of proposition (2.9) for some $\theta \in]0, 1[$, hence it is well posed in $C([-T_0, T_0], H^1)$. Thus, (2) is well-posed in $U(T)(v_0, v_1) + C([-T_0, T_0], H^1)$. □

Definition 2.13. Let T_N be an increasing sequence of \mathbb{R} going to $+\infty$ and let

$$E = \limsup E(T_N) .$$

Proposition 2.14. *The set E is of full μ -measure.*

Proof. Indeed, using Fatou's lemma,

$$\mu(E^c) = \mu(\liminf E(T_N)^c) \leq \liminf \mu(E(T_N)^c) = 0 .$$

□

Theorem 2.15. *Let $v_0, v_1 \in E$, the equation on the sphere (1) with initial datum (v_0, v_1) has a unique global solution in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1)$.*

Proof. Let $T \geq 0$. As the sequence T_N is increasing toward ∞ there exists N_0 such that for all $N \geq N_0$,

$$T_N \geq T_{N_0} \geq T .$$

Since $E = \limsup E(T_N)$, for all N_1 there exists $N \geq N_1$ such that $v_0, v_1 \in E(T_N)$. With $N_1 = N_0$ there exists $N \geq N_0$ such that

$$T_N \geq T \text{ and } v_0, v_1 \in E(T_N) .$$

Hence the equation has a unique solution on $U(\tau)(v_0, v_1) + C([-T_N, T_N], H^1)$ and thus in $U(\tau)(v_0, v_1) + C([-T, T], H^1)$. Therefore, this property holding for all time T , the equation has a unique solution in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1)$. □

3 Reduction to the sphere and almost sure solutions on the Euclidian space

In this section, the problem on the Euclidian space is reduced thanks to the Penrose transform to the problem on the sphere. The existence of solution for the Cauchy problem with initial data on a suitable space is derived in this way. Note that for all the sequel $\sigma = 0$.

3.1 Penrose transform and reduction to the sphere

As a basis of L^2 uniformly bounded in L^p is required to use the techniques developed by N. Burq and N. Tzvetkov in [2] and according to [1], the problem needs to be reduced to the sphere. For that, the Penrose transform seems appropriate, since it turns the d'Alembertian of \mathbb{R}^3 into the d'Alembertian of S^3 added to the identity on distributions.

Definition 3.1 (Penrose Transform on the variables). For all $t \in \mathbb{R}$ and $r \in \mathbb{R}^+$, define $T(t, r)$, $R(t, r)$, $R_0(r)$, $\Omega(t, r)$ and $\Omega_0(r)$ as :

$$T = \text{Arctan}(t+r) + \text{Arctan}(t-r), \quad R = \text{Arctan}(t+r) - \text{Arctan}(t-r), \quad R_0(r) = R(0, r) = 2\text{Arctan}(r),$$

$$\Omega(t, r) = \cos T + \cos R = \frac{2}{\sqrt{(1+(t+r)^2)(1+(t-r)^2)}} \text{ and } \Omega_0(r) = \Omega(0, r) = \frac{2}{1+r^2}.$$

Proposition 3.2. *The map*

$$t, r, \omega \in \mathbb{R} \times \mathbb{R}^+ \times S^2 \mapsto T(t, r), R(t, r), \omega$$

is a bijection from $\mathbb{R} \times \mathbb{R}^3$ to $S = \{(T, R, \omega) \mid \cos T + \cos R > 0\}$ and its inverse is given by

$$T, R, \omega \mapsto t = \frac{\sin T}{\cos T + \cos R}, \quad r = \frac{\sin R}{\cos T + \cos R}, \quad \omega.$$

See [7, 4] for the proof.

Remark 3.1. *The map $r, \omega \mapsto 2\text{Arctan}(r), \omega$ is a bijection from \mathbb{R}^3 to S^3 deprived of one of its poles, $R_0(r) = 2\text{Arctan}(r) \in [0, \pi[$ being the third angle describing a point in S^3 .*

Definition 3.3 (Penrose Transform on distributions). Let f be a distribution on $\mathbb{R} \times \mathbb{R}^3$ and (f_0, f_1) be a pair of distributions on \mathbb{R}^3 . Define then $v = \text{PT}(f)$ the distribution on S and $(v_0, v_1) = \text{PT}_0(f_0, f_1)$ the pair of distributions on S^3 deprived of one of its poles such that

$$v(T, R, \omega) = f\left(\frac{\sin T}{\cos T + \cos R}, \frac{\sin R}{\cos T + \cos R}, \omega\right)(\cos T + \cos R)^{-1}$$

and

$$v_0(R, \omega) = \frac{f_0(\tan(R/2), \omega)}{1 + \cos R}, \quad v_1(R, \omega) = \frac{f_1(\tan(R/2), \omega)}{(1 + \cos R)^2}.$$

Remark 3.2. The definition of PT_0 may appear a little awkward but the idea hidden behind the notations is that f solves the cubic non linear wave equation with initial datum (g_0, g_1) if an extension of $PT(f)$ solves the equation of the first section with initial datum an extension to S^3 of $PT_0(g_0, g_1)$.

Definition 3.4. Let u be a distribution on $\mathbb{R} \times S^3$ and v_0, v_1 two distributions on S^3 , the inverse Penrose transform is given by :

$$PT^{-1}u(t, r, \omega) = \Omega(t, r)u(\text{Arctan}(t+r) + \text{Arctan}(t-r), \text{Arctan}(t+r) - \text{Arctan}(t-r), \omega),$$

which depends only on the restriction of u to S and the inverse Penrose transform at time $t = 0 \Leftrightarrow t = 0$ by

$$PT_0^{-1}(r, \omega)(v_0, v_1) = (\Omega_0(r)v_0(2\text{Arctan}(r), \omega), \Omega_0^2(r)v_1(2\text{Arctan}(r), \omega)),$$

which depends only on the restriction on S^3 deprived of one of its poles of v_0, v_1 .

Lemma 3.5. If u solves the problem

$$\begin{cases} (\partial_T^2 + 1 - \Delta_{S^3})u + |u|^2u = 0 \\ (u|_{T=0}, \partial_T u|_{T=0}) = v_0, v_1 \end{cases} \quad (4)$$

then the map f defined as the inverse Penrose transform of u restricted to S , that is

$$f = PT^{-1}(u)$$

solves the problem :

$$\begin{cases} (\partial_t^2 - \Delta_{\mathbb{R}^3})f + |f|^2f = 0 \\ f|_{t=0} = g_0 \quad \partial_t f|_{t=0} = g_1 \end{cases} \quad (5)$$

where

$$(g_0, g_1) = PT_0^{-1}(v_0, v_1).$$

Proof. The fact that the Penrose transform sends the action of $\partial_t^2 - \Delta_{\mathbb{R}^3}$ on $\mathbb{R} \times \mathbb{R}^3$ onto the action of $\Omega^3(\partial_T^2 + 1 - \Delta_{S^3})$ on S is known and the proof can be found in [7]. Thus, on S

$$((\partial_t^2 - \Delta_{\mathbb{R}^3})f + |f|^2f)(t, r, \omega) = \Omega^3(\partial_T^2 + 1 - \Delta_{S^3})PT(f) + \Omega^3|PT(f)|^2PT(f)(T(t, r), R(t, r), \omega) = 0.$$

What is more, $T = 0 \Leftrightarrow t = 0$,

$$g_0 = f(t = 0) = \Omega_0 u(T = 0) = \Omega_0 u(R_0(r))$$

and

$$\begin{aligned} g_1 &= \partial_t f(t = 0) = (\partial_t \Omega)(t = 0)u(0, R_0(r)) + \Omega_0 \partial_t T(t = 0) \partial_T u + \Omega_0 \partial_t R(t = 0) \partial_R u \\ &= \Omega_0(r)^2 \partial_T u = \Omega_0^2 v_1(R_0(r)). \end{aligned}$$

□

3.2 Properties of the change of variable

In this subsection, the properties of the change of variables involved in the Penrose transform is studied, in particular what it implies on operators and norms.

Definition 3.6. Let Ψ be the change of variable corresponding to the Penrose transform at time $T = 0$, that is to say :

$$\Psi(v)(r, \omega) = v(2\text{Arctan}(r), \omega) .$$

Proposition 3.7. *This change of variable satisfies :*

- for all v, w , $\int v(R, \omega)w(R, \omega) \sin^2 R d\omega dR = \int \Psi(v)(r, \omega)\Psi(w)(r, \omega)r^2 \left(\frac{2}{1+r^2}\right)^3 dr$,
- for all v , $\int |v|^p \sin^2 R dR = \int |\Psi(v)|^p r^2 \left(\frac{2}{1+r^2}\right)^3 dr$,
- $\Psi(\Delta_{S^3} v) = \left(\frac{1+r^2}{2}\right)^2 \Delta_{\mathbb{R}^3} \Psi(v) - \frac{1+r^2}{2} r \partial_r \Psi(v)$.

Proof. The proof comes from the facts that :

- $dR = \frac{2}{1+r^2} dr$,
- $\sin R = \frac{2r}{1+r^2}$ and
- $\tan R = \frac{2r}{1-r^2}$.

Hence, to do the change of variable in the integrals, one can use :

$$\sin^2 R dR = \left(\frac{2}{1+r^2}\right)^3 r^2 dr .$$

The computation of the change of variable on the Laplace-Beltrami operator is quite similar:

$$\begin{aligned} \Psi(\partial_R v) &= (\partial_r R)^{-1} \partial_r \Psi(v) \\ \Psi(\sin^2(R) \partial_R v) &= \frac{2r^2}{1+r^2} \partial_r \Psi(v) \\ \Psi(\partial_R \sin^2(R) \partial_R v) &= (\partial_r R)^{-1} \partial_r \Psi(\sin^2 R \partial_R v) \\ \Psi(\partial_R \sin^2(R) \partial_R v) &= -\frac{2r^3}{1+r^2} \partial_r \Psi(v) + \partial_r \left(r^2 \partial_r \Psi(v) \right) \\ \Psi\left(\frac{1}{\sin^2 R} \partial_R \sin^2(R) \partial_R v\right) &= -\frac{1+r^2}{2} r \partial_r \Psi(v) + \left(\frac{1+r^2}{2}\right)^2 \frac{1}{r^2} \partial_r (r^2 \partial_r \Psi(v)) \\ \Psi(\Delta_{S^3} v) &= \left(\frac{1+r^2}{2}\right)^2 \Delta_{\mathbb{R}^3} \Psi(v) - \frac{1+r^2}{2} r \partial_r \Psi(v). \end{aligned}$$

□

Definition 3.8. Let f_0, f_1 be the inverse Penrose transform at time $T = 0$ of (u_0, u_1) and $g_{n,k}, h_{n,k}$ the inverse Penrose transform at time $T = 0$ of $e_{n,k}, e_{n,k}$, that is to say :

$$f_0 = \sum_{n,k} u_0^{n,k} a_{n,k} g_{n,k} , \quad f_1 = \sum_{n,k} u_1^{n,k} b_{n,k} h_{n,k} .$$

Proposition 3.9. *The $g_{n,k}$ are the eigenfunctions of $H_0 = \left(\frac{1+r^2}{2}\right)^2 \Delta_{\mathbb{R}^3} + \frac{1+r^2}{2} r \partial_r + \frac{3+r^2}{2}$ with eigenvalues $1 - n^2$ and the $h_{n,k}$ are the eigenfunctions of $H_1 = \left(\frac{1+r^2}{2}\right)^2 \Delta_{\mathbb{R}^3} + 3\frac{1+r^2}{2} r \partial_r + 6\frac{1+r^2}{2}$ with eigenvalues $1 - n^2$, $n \geq 1$.*

Proof. As

$$g_{n,k} = \frac{2}{1+r^2} \Psi(e_{n,k})$$

they are the eigenfunctions of the operator H_0 such that

$$H_0(g) = \frac{2}{1+r^2} \Psi \left(\Delta_{S^3} \Psi^{-1} \left(\frac{1+r^2}{2} g \right) \right).$$

It remains to compute H_0 .

$$H_0 g = \left(\frac{1+r^2}{2} \right)^2 \Delta_{\mathbb{R}^3} g + \frac{1+r^2}{2} r \partial_r g + \frac{3+r^2}{2} g$$

As

$$h_{n,k} = \left(\frac{2}{1+r^2} \right)^2 e_{n,k}$$

they are the eigenfunctions of H_1 such that

$$\begin{aligned} H_1(h) &= \left(\frac{2}{1+r^2} \right)^2 \Psi \left(\Delta_{S^3} \Psi^{-1} \left(\frac{1+r^2}{2} \right)^2 h \right) \\ H_1 h &= \left(\frac{1+r^2}{2} \right)^2 \Delta_{\mathbb{R}^3} h + 3 \frac{1+r^2}{2} r \partial_r h + 6 \frac{1+r^2}{2} h. \end{aligned}$$

□

Corollary 3.10. *The maps f_0 and f_1 such that $(f_0, f_1) = PT_0^{-1}(u_0, u_1)$ satisfies*

$$\begin{aligned} \|u_0\|_{W^{s,p}(S^3)} &= \left\| \left(\frac{2}{1+r^2} \right)^{3/p-1} (1-H_0)^{s/2} f_0 \right\|_{L^p(\mathbb{R}^3)} \\ \|u_1\|_{W^{s,p}(S^3)} &= \left\| \left(\frac{2}{1+r^2} \right)^{3/p-2} (1-H_1)^{s/2} f_1 \right\|_{L^p(\mathbb{R}^3)}. \end{aligned}$$

Proof. First, do the change of variable in the L^p -norm :

$$\|u_0\|_{W^{s,p}} = \left\| \left(\frac{2}{1+r^2} \right)^{3/p} \Psi((1-\Delta_{S^3})^{s/2} u_0) \right\|_{L^p}.$$

Then, compute $\Psi((1-\Delta_{S^3})^{s/2} u_0)$:

$$\begin{aligned} \Psi((1-\Delta_{S^3})^{s/2} u_0) &= \Psi \left(\sum_{n,k} n^s u_0^{n,k} a_{n,k} e_{n,k} \right) = \frac{1+r^2}{2} \sum_{n,k} n^s u_0^{n,k} a_{n,k} g_{n,k} \\ &= \frac{1+r^2}{2} \sum_{n,k} (1-H_0)^{s/2} u_0^{n,k} a_{n,k} g_{n,k} = \frac{1+r^2}{2} (1-H_0)^{s/2} f_0. \end{aligned}$$

In the end, it comes :

$$\|u_0\|_{W^{s,p}} = \left\| \left(\frac{2}{1+r^2} \right)^{3/p-1} (1-H_0)^{s/2} f_0 \right\|_{L^p} .$$

The second equality is proved the same way. \square

3.3 Spaces of definition of the initial data

Considering the results of the previous subsection, the choice of the random variable $a_{n,k}$ and $b_{n,k}$ will be made such that the initial datum u_0, u_1 of the equation reduced to the sphere is a Gaussian vector, in order to state which norms of u_0 and u_1 and then of the initial datum of the wave equation on the Euclidian space are almost surely finite or infinite.

In this subsection, suppose that $a_{n,k}$ and $b_{n,k}$ not only satisfy the Gaussian large deviation estimate, but that they are Gaussian. To ensure that the initial datum is almost surely not into some spaces, Fernique's theorem should be used :

Theorem 3.11 (Fernique, 1974). *Let X be a Gaussian vector with value into a Banach space B and N a pseudo-norm on B (a pseudo-norm has the same properties as a norm except that ∞ is one of its possible value), then for all $p \geq 1$ if the mean value of $N(X)^p$ is infinite, then $N(X)$ is almost surely ∞ :*

$$E(N(X)^p) = \infty \Rightarrow P(N(X) = \infty) = 1 .$$

For the proof and furthermore, see [6].

Proposition 3.12. *The initial datum u_0, u_1 is almost surely not in $H^s \times H^{s-1}$ for all $s > 0$.*

Proof. Use Fernique's theorem with $B = L^2$, $p = 2$ and N being the H^s norm and X either u_0 or u_1 . As

$$E(\|u_0\|_{H^s}^2) = \sum_{n,k} (u_0^{n,k})^2 n^{2s} ,$$

$$E(\|u_1\|_{H^{s-1}}^2) = \sum_{n,k} (u_1^{n,k})^2 n^{2(s-1)} ,$$

either one of this series diverges and u_0 and u_1 are pseudo Gaussian vectors, it comes that almost surely

$$\|u_0\|_{H^s} = \infty$$

or almost surely

$$\|u_1\|_{H^{s-1}} = \infty .$$

\square

Considering the remarks on the norms of the initial datum (f_0, f_1) wrt the ones of (u_0, u_1) in the previous subsection, the author believes that the initial datum f_0, f_1 of the cubic non linear wave equation belongs almost surely to $L^2 \times H^{-1}$ with weight $\frac{1}{\sqrt{1+r^2}}$ but is almost surely not in $H^s \times H^{s-1}$ for all $s > 0$.

Nevertheless, the proof would require the equivalence between the norms

$$\left\| \left(\frac{2}{1+r^2} \right)^{1/2} (1-H_0)^{s/2} \cdot \right\|_{L^2(\mathbb{R}^3)} \text{ and } \left\| \left(\frac{2}{1+r^2} \right)^{1/2-s} (1-\Delta_{\mathbb{R}^3})^{s/2} \cdot \right\|_{L^2(\mathbb{R}^3)}$$

in the one hand and

$$\left\| \left(\frac{2}{1+r^2} \right)^{-1/2} (1-H_1)^{s/2} \cdot \right\|_{L^2(\mathbb{R}^3)} \text{ and } \left\| \left(\frac{2}{1+r^2} \right)^{-1/2-s} (1-\Delta_{\mathbb{R}^3})^{s/2} \cdot \right\|_{L^2(\mathbb{R}^3)}$$

on the other hand.

Definition 3.13. Let $\mathcal{H}_0^s(\mathbb{R}^3)$ and $\mathcal{H}_1^s(\mathbb{R}^3)$ be the topological spaces defined by the norms

$$\|g\|_{\mathcal{H}_0^s(\mathbb{R}^3)} = \left\| \left(\frac{2}{1+r^2} \right)^{1/2} (1-H_0)^s g \right\|_{L^2}$$

and

$$\|g\|_{\mathcal{H}_1^s(\mathbb{R}^3)} = \left\| \left(\frac{2}{1+r^2} \right)^{-1/2} (1-H_1)^s g \right\|_{L^2} .$$

Proposition 3.14. *The set $F = PT_0^{-1}(E)$ is almost surely included in $\mathcal{H}_0^0 \times \mathcal{H}_1^{-1}$ and almost surely disjoint from $\mathcal{H}_0^s \times \mathcal{H}_1^{s-1}$ for all $s > 0$.*

Setting $f_0, f_1 = PT_0^{-1}(u_0, u_1)$, the random variable which is used as initial datum of the cubic NLW on \mathbb{R}^3 and ν the image measure of μ under PT_0 , the set F satisfies $\nu(F^c) = 0$, which means that there exists ν almost surely a solution of the cubic NLW.

4 Uniqueness of the solution and scattering

In this section, the uniqueness of the solution, alongside with some scattering properties is proved.

4.1 Uniqueness

Theorem 4.1. *Let $g_0, g_1 \in PT_0^{-1}(E)$. The Cauchy problem*

$$\begin{cases} \partial_t^2 f - \Delta f + f^3 = 0 \\ f|_{t=0} = g_0, & \partial_t f|_{t=0} = g_1 \end{cases}$$

has a unique solution in $L(t)(g_0, g_1) + C(\mathbb{R}, H^1(\mathbb{R}^3))$ where $L(t)$ is the flow of $\partial_t^2 - \Delta = 0$.

Proof. Let $v_0, v_1 \in E$ such that (g_0, g_1) is the inverse Penrose transform of v_0, v_1 , let u be the solution of the equation on the sphere with initial datum v_0, v_1 . Let f be the Penrose transform of u restricted to S , this map f satisfies the Cauchy problem with initial datum g_0, g_1 , which gives the existence of the solution. Prove now that this solution is unique.

Lemma 4.2. *It appears that :*

$$PT^{-1}U(T)(v_0, v_1) = (L(t)(g_0, g_1)) .$$

Proof. The map $w = U(T)(v_0, v_1)$ satisfies

$$\partial_T^2 w + (1 - \Delta_{S^3})w = 0$$

with initial datum v_0, v_1 . Hence its inverse Penrose transform h satisfies

$$\partial_t^2 h - \Delta_{\mathbb{R}^3} h = \Omega^3 \Psi \left(\partial_T^2 h + (1 - \Delta_{S^3})h \right) = 0$$

with initial datum g_0, g_1 , that is, $h = L(t)(g_0, g_1)$. \square

Then, let $g = f - L(t)(g_0, g_1)$, g is the reverse Penrose transform of $v = u - U(T)(v_0, v_1)$ and is the solution of

$$\partial_t^2 g - \Delta g + (L(t)(g_0, g_1) + g)^3 = 0$$

with initial datum 0, 0, hence it is a fixed point of

$$\phi(g) = \int_0^t \frac{\sin(\sqrt{-\Delta}(t-s))}{\sqrt{-\Delta}} (L(t)(g_0, g_1) + g) ds .$$

Lemma 4.3. *Let $w \in L^q([-\pi, \pi] \times S^3)$, $q \geq 4$ and h its reverse Penrose transform, then*

$$\|h\|_{L^q(\mathbb{R} \times \mathbb{R}^3)} \leq C \|w\|_{L^q([-\pi, \pi] \times S^3)} .$$

Proof. Computing the change of variable $(T, R) = (\text{Arctan}(t+r) + \text{Arctan}(t-r), \text{Arctan}(t+r) - \text{Arctan}(t-r))$ leads to:

$$\int_{\mathbb{R} \times \mathbb{R}^3} |h(t, r, \omega)|^q r^2 dr dt d\omega = \int_{\Omega > 0} |\Omega w(R, T, \omega)|^q \Omega^{-4} \sin^2 R dR dT d\omega .$$

With $q \geq 4$ and $\Omega = \cos T + \cos R$ being bounded by 2,

$$\|h\|_{L^q(\mathbb{R} \times \mathbb{R}^3)} \leq C \|w\|_{L^q} .$$

\square

Therefore, $L(t)(g_0, g_1)$, g and so f belong to $L^6(\mathbb{R} \times \mathbb{R}^3)$. Indeed,

$$\begin{aligned} \|L(t)(g_0, g_1)\|_{L^6(\mathbb{R} \times \mathbb{R}^3)} &\leq C \|U(T)(v_0, v_1)\|_{L^6} \\ &\leq C \left(\int_{-\pi}^{\pi} (1 + T^{\delta_3})^6 \right)^{1/6} \|(1 + T^{\delta_3})^{-1} U(T)(v_0, v_1)\|_{L_T^3, L^6(S^3)} < \infty , \\ \|g\|_{L^6} &\leq C \|v\|_{L^6} \leq C (2\pi)^{1/6} \|v\|_{L_T^\infty, L^6} \leq C \|v\|_{L_T^\infty, H^1} < \infty . \end{aligned}$$

Lemma 4.4. *The map g belongs to $C(\mathbb{R}, H^1(\mathbb{R}^3))$ and $\partial_t g \in C(\mathbb{R}, L^2(\mathbb{R}^3))$.*

Proof. The map g satisfies

$$g = - \int_0^\tau \frac{\sin(\tau - s) \sqrt{-\Delta}}{\sqrt{-\Delta}} (L(s)(g_0, g_1) + g)^3 ds .$$

Hence,

$$\partial_t g = - \int_0^\tau \cos((\tau - s) \sqrt{-\Delta}) (L(s)(g_0, g_1) + g)^3 ds ,$$

$$\|\partial_t g\|_{L^2(\mathbb{R}^3)} \leq \int_0^\tau \|L(s)(g_0, g_1) + g\|_{L^6(\mathbb{R}^3)}^3 ds \leq \sqrt{\tau} \|L(s)(g_0, g_1) + g\|_{L^6(\mathbb{R} \times \mathbb{R}^3)}^3$$

$$\|\partial_t g\|_{L^\infty([0, t], L^2(\mathbb{R}^3))} \leq \sqrt{t} \|L(s)(g_0, g_1) + g\|_{L^6(\mathbb{R} \times \mathbb{R}^3)}^3 .$$

Therefore, as initially, $g(\tau = 0) = 0$, g belongs to $C(\mathbb{R}, L^2)$.

Besides,

$$\|g\|_{\dot{H}^1} \leq \int_0^\tau \|L(s)(g_0, g_1) + g\|_{L^6(\mathbb{R}^3)}^3 ds .$$

Hence,

$$\|g\|_{\dot{H}^1} \leq \sqrt{t} \|L(s)(g_0, g_1) + g\|_{L^6(\mathbb{R} \times \mathbb{R}^3)}^3 .$$

Therefore, g belongs to $C([0, t], H^1(\mathbb{R}^3))$. □

Prove now the uniqueness of the solution in $L(t)(g_0, g_1) + C([0, t], H^1(\mathbb{R}^3))$. Let f_2, f_3 be two solutions of the cubic wave equation with initial datum g_0, g_1 , let $h = f_2 - f_3$. The map h satisfies :

$$\partial_t^2 h - \Delta h + f_2^3 - f_3^3 = 0 .$$

Remark that h is in H^1 and $\partial_t h$ is in L^2 . Let

$$H(t)^2 = \int (\partial_t h)^2 + \int h(1 - \Delta)h .$$

$$2H'(t)H(t) = -2 \int (\partial_t h) (-h + f_2^3 - f_3^3) = -2 \int (\partial_t h) (-h + h(f_2^2 + f_2 f_3 + f_3^2))$$

$$H'(t) \leq C \|h\|_{L^2} + C \|h\|_{L^6} (\|f_2\|_{L^6}^2 + \|f_3\|_{L^6}^2)$$

As $H(0) = 0$ and

$$\int_0^t (\|f_2\|_{L^6}^2 + \|f_3\|_{L^6}^2) ds \leq |t|^{2/3} (\|f_2\|_{L_{t,x}^6}^2 + \|f_3\|_{L_{t,x}^6}^2) ,$$

by Gronwall lemma, $H(t) = 0$ for all time t , which proves the uniqueness. □

4.2 Scattering property

Finally, with those particular initial data for the wave equation, it satisfies a scattering property. More precisely, when t goes to $\pm\infty$ the solution tends to behave like the solution of the linearized around 0 solution of the equation with same initial datum. This property does not result from a scattering property of the wave equation on the sphere. Indeed, it is the fact that the Penrose transform divides the solution by something that behaves like $\frac{1}{t^2}$ that ensures scattering.

Theorem 4.5. *Let $q \in]\frac{18}{5}, 6]$, $(g_0, g_1) \in PT_0^{-1}(E)$, $f(t)$ the solution of the cubic wave equation with initial datum g_0, g_1 .*

There exists a constant C depending on the initial datum such that

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} \leq \frac{C}{(1+t^2)^{1/6}}.$$

Proof. Let $v_0, v_1 \in E$ such that $(g_0, g_1) = PT_0^{-1}(v_0, v_1)$ and u the solution of (1) with initial datum v_0, v_1 .

The map u satisfies :

$$u(T) - U(T)(v_0, v_1) = - \int_0^T \frac{\sin((T-\tau)\sqrt{1-\Delta})}{\sqrt{1-\Delta}} (u^3(\tau)) d\tau.$$

Taking the inverse of the Penrose transform of this equality leads to :

$$f(t) - L(t)(g_0, g_1) = -\Omega(t, r) \left(\int_0^{T(t,r)} \frac{\sin(T-\tau)\sqrt{1-\Delta}}{\sqrt{1-\Delta}} (u^3(\tau)) d\tau \right) (R(t, r)).$$

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} \leq \left\| \frac{\Omega}{2\Omega^{2/3}(1+t^2+r^2)^{1/6}} \right\|_{L^p}$$

$$\left\| \left(\int_0^{T(t,r)} \frac{\sin(T-\tau)\sqrt{1-\Delta}}{\sqrt{1-\Delta}} (u^3(\tau)) d\tau \right) (R(t, r)) 2\Omega^{2/3}(1+t^2+r^2)^{1/6} \right\|_{L^6}$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{6}$ ($q \leq 6$).

Let

$$A = \left\| \frac{\Omega}{2\Omega^{2/3}(1+t^2+r^2)^{1/6}} \right\|_{L^p}$$

and

$$B = \left\| \left(\int_0^{T(t,r)} \frac{\sin(T-\tau)\sqrt{1-\Delta}}{\sqrt{1-\Delta}} (u^3(\tau)) d\tau \right) (R(t, r)) 2\Omega^{2/3}(1+t^2+r^2)^{1/6} \right\|_{L^6}.$$

Apply the change of variable $R = \text{Arctan}(t+r) - \text{Arctan}(t-r)$ (t is fixed) in B .

$$dR = \Omega^2 2(1+t^2+r^2) dr$$

$$\sin^2 R dR = \Omega^4 2(1+t^2+r^2) r^2 dr$$

$$B = \left\| \int_0^{T(t,R)} \frac{\sin(T-\tau)\sqrt{1-\Delta}}{\sqrt{1-\Delta}} (u^3(\tau)) d\tau \right\|_{L_{S^3}^6}$$

$$\begin{aligned}
B &= \left\| \int_{-\pi}^{\pi} 1_{\tau \leq T(t,R)} \frac{\sin(T-\tau) \sqrt{1-\Delta}}{\sqrt{1-\Delta}} (u^3(\tau)) d\tau \right\|_{L^6} \\
B &\leq \int_{-\pi}^{\pi} \left\| 1_{\tau \leq T(t,R)} \frac{\sin(T-\tau) \sqrt{1-\Delta}}{\sqrt{1-\Delta}} (u^3(\tau)) \right\|_{L^6} d\tau \\
B &\leq \int_{-\pi}^{\pi} \left\| \frac{\sin(T-\tau) \sqrt{1-\Delta}}{\sqrt{1-\Delta}} (u^3(\tau)) \right\|_{L^6} d\tau \\
B &\leq C \|u\|_{L^3_{T \in [-\pi, \pi]}}^3 \cdot L^6.
\end{aligned}$$

To bound A remark that $\Omega \leq \frac{2}{\sqrt{1+t^2}}$.

$$A \leq \frac{1}{(1+t^2)^{1/6}} \|\Omega^{1/3}\|_{L^p} = \frac{1}{(1+t^2)^{1/6}} \|\Omega\|_{L^{p/3}}^{1/3}.$$

As $q > \frac{18}{5}$,

$$\frac{p}{3} = \frac{2q}{6-q} > 3$$

which ensures that $\Omega \in L^{p/3}$ and bounded uniformly in t .

Finally,

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} \leq AB \leq \frac{C}{(1+t^2)^{1/6}}.$$

□

References

- [1] N. Burq and G. Lebeau, *Injectons de sobolev probabilistes*, preprint.
- [2] N. Burq and N. Tzvetkov, *Global solutions of super-critical regularity*, preprint.
- [3] Nicolas Burq and Nikolay Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory*, Invent. Math. **173** (2008), no. 3, 449–475.
- [4] D. Christodoulou, *Global solutions of non linear hyperbolic equations for small initial data*, Comm. Pure. Appl. Math., vol. 39 (1986), pp 267-282.
- [5] A-S. de Suzzoni, *Large data low regularity scattering results for the wave equation on the euclidian space*, preprint.
- [6] X. Fernique, *Régularité des trajectoires des fonctions aléatoires gaussiennes*, Ecole d'été St. Flour.
- [7] N. Tzvetkov, *Remark on the Null-Condition for the Nonlinear Wave Equation*, Bollettino U.M.I. **8** (2000), no. 1-B, 135–145.